MATHEMATICS SPECIALIST

MAWA Year 12 Examination 2016

Calculator-free

Marking Key

The release date for this exam and marking scheme is

- **the end of week 1 of term 4, 2016**

Question 1 **(8 marks)**

(a) If $(a-3i)^2 = -5-bi$ find the values of a and b, where a and b are real constants. (3 marks)

Solution
$(a-3i)^2 = a^2 - 9 - 6ai$ Equating the real parts: $a^2 - 9 = -5 \Rightarrow a = \pm 2$ Equating the imaginary parts: $-6a = -b$: So, for $a = 2,\ b = 12$ and for $a = -2,\ b = -12$
Specific behaviours
✓ correctly expands $(a-3i)^2$ ✓ equates real and imaginary parts ✓ correctly states corresponding values of a and b

(b) The complex number $z = 1-\sqrt{3}i$ is transformed to its reciprocal $\frac{1}{1-\sqrt{3}i}$.

(i) What is the reciprocal of z in the form $a+bi$? (2 marks)

Solution
$\frac{1}{1-\sqrt{3}i} = \frac{1}{1-\sqrt{3}i} \cdot \frac{1+\sqrt{3}i}{1+\sqrt{3}i} = \frac{1+\sqrt{3}i}{4} = \frac{1}{4}(1+\sqrt{3}i)$
Specific behaviours
✓ multiplies by $\frac{\bar{z}}{\bar{z}}$ ✓ simplifies to arrive at the correct result

(ii) State the reciprocal of $z = 1-\sqrt{3}i$ in polar form. (2 marks)

Solution
$mod\left(\frac{1}{1-\sqrt{3}i}\right) = \frac{1}{2}$ and $arg\left(\frac{1}{1-\sqrt{3}i}\right) = arg\left(\frac{1+\sqrt{3}i}{4}\right) = arctan\left(\sqrt{3}\right) = \frac{\pi}{3}$ $\therefore \frac{1}{1-\sqrt{3}i} = \frac{1}{2}cis\left(\frac{\pi}{3}\right)$ in polar form.
Specific behaviours
✓ determines $arg\left(\frac{1}{1-\sqrt{3}i}\right) = \frac{\pi}{3}$ ✓ correctly states $\frac{1}{z}$ in polar form

(c) Given z is a complex number, express the modulus and argument of $\frac{1}{z}$ in terms of $\operatorname{mod} z$ and $\arg z$. (1 mark)

Solution
$mod\left(\frac{1}{z}\right)=\frac{1}{mod\ z}$ or $\left\lvert\frac{1}{z}\right\rvert=\frac{1}{\lvert z\rvert}$ $arg\left(\frac{1}{z}\right)=-arg(z)$
Specific behaviours
✓ states the correct relationship for the modulus and the argument

Question 2 **(8 marks)**

Let $f(x)=\frac{1}{x-3}$ and $g(x)=2x-1$. Determine the following:

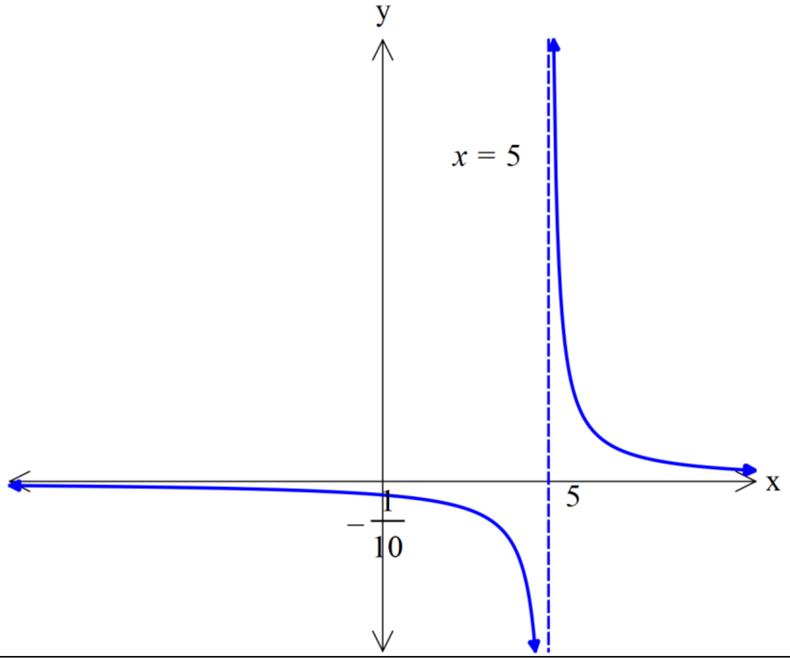
(a) $f\circ g(x)$ and its natural domain. (2 marks)

Solution
$f\circ g(x)=\frac{1}{2x-1-3}=\frac{1}{2x-4}$ $x\neq 2$ Domain: $\mathbb{R}-\{2\}$ or $\{x:x\neq 2, x\in\mathbb{R}\}$
Specific behaviours
✓ states rule of composite ✓ states natural domain

(b) $f\circ g(x-3)$ and its natural domain and range. (3 marks)

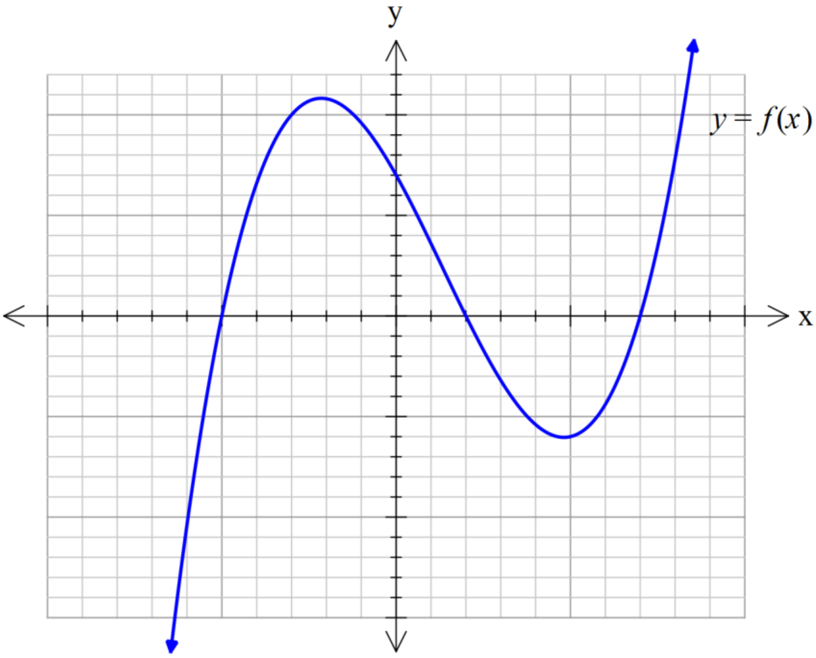
Solution
$f\circ g(x)=\frac{1}{2x-4}$ $f\circ g(x-3)=\frac{1}{2(x-3)-4}=\frac{1}{2x-10}$ $x\neq 5$ Domain: $\mathbb{R}-\{5\}$ or $\{x:x\neq 5, x\in\mathbb{R}\}$ Range: $\mathbb{R}-\{0\}$ or $\{y:y\neq 0, y\in\mathbb{R}\}$
Specific behaviours
✓ states simplified rule of composite ✓ states natural domain ✓ states range

(c) Sketch $y = f \circ g(x-3)$ on the axes below showing all major features. (3 marks)

Solution

Specific behaviours
✓ shows graph to be in two separate parts ✓ shows asymptotes $x=5$ and $y=0$ ✓ labels y intercept

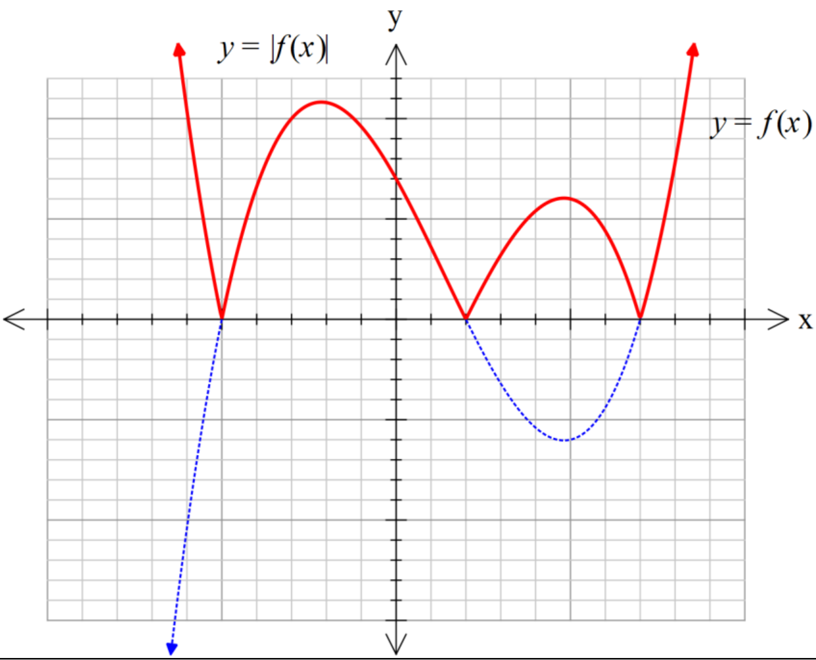
Question 3 **(5 marks)**

Consider the function f as graphed below:

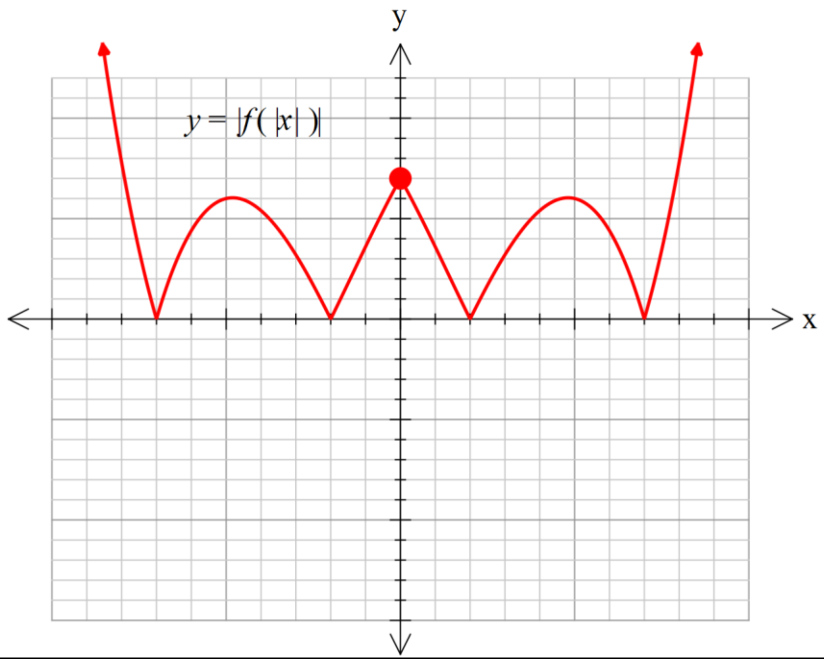


On the set of axes provided, sketch the new curve given that the dotted curve is $y = f(x)$.

(a) Sketch $y = |f(x)|$. (2 marks)

Solution

Specific behaviours
✓ leaves *f(x)* unchanged for positive y values ✓ reflects in x axis for where y is negative

(b) On the axes below, sketch $y = \left| f\left(|x|\right) \right|$. (3 marks)

Solution

Specific behaviours
✓ shows all intercepts ✓ shows that left side is a reflection of right side ✓ all y values greater than or equal to zero

Question 4 **(11 marks)**

Determine the following integrals.

(a) $\int \frac{x}{7x^2+1}\,dx$ (2 marks)

Solution
$\int \frac{x}{7x^2+1}\,dx$ $\frac{1}{14}\int \frac{14x}{7x^2+1}dx = \frac{1}{14}\ln\left(7x^2+1\right)+c$
Specific behaviours
✓ recognises that numerator is proportional to derivative of numerator ✓ uses natural log with a constant

(b) $\int \cos^2(5x)\,dx$ (3 marks)

Solution
$\int \cos^2(5x)\,dx$ $=\int \frac{\cos(10x)+1}{2}dx$ $=\frac{1}{20}\sin(10x)+\frac{1}{2}x+c$
Specific behaviours
✓ uses double angle formula for cosine ✓ integrates one term correctly ✓ integrates all terms correctly

Determine the following integrals with the given substitution.

(c) $\int \frac{15x+1}{\sqrt{1-5x}}\,dx \quad u=1-5x$ (3 marks)

Solution
$\int\left(\frac{15x+1}{\sqrt{1-5x}}\right)dx \quad u=1-5x$ $\int\left(\frac{15\frac{1-u}{5}+1}{\sqrt{u}}\times\frac{-1}{5}\right)du=\frac{-1}{5}\int(4-3u)u^{-\frac{1}{2}}du$ $\frac{-1}{5}\int 4u^{-\frac{1}{2}}-3u^{\frac{1}{2}}du=\frac{-1}{5}\left(8u^{\frac{1}{2}}-2u^{\frac{3}{2}}\right)+c$ $=\frac{-8}{5}(1-5x)^{\frac{1}{2}}+\frac{2}{5}(1-5x)^{\frac{3}{2}}+c$
Specific behaviours
✓ changes variable to u in integral ✓ antidifferentiates with respect to u ✓ expresses in terms of x

(d) $\int_0^{\frac{\pi}{2}} 5\sin^7(3x)\cos(3x)\,dx \quad u=\sin(3x)$ (3 marks)

Solution
$\int_0^{\frac{\pi}{2}} 5\sin^7(3x)\cos(3x)\,dx \quad u=\sin(3x)$ $\int_0^{-1} 5u^7\cos(3x)\frac{1}{3\cos(3x)}du=\frac{5}{3}\int_0^{-1}u^7du=\frac{5}{3}\left[\frac{u^8}{8}\right]_0^{-1}=\frac{5}{24}$
Specific behaviours
✓ changes variable to u in integral ✓ changes limits to u values ✓ determines definite integral

Question 5 **(8 marks)**

Consider the following system of linear equations:

$$\begin{aligned} x + 2y + 3z &= 2 \\ 3x + 7y + 11z &= 6 \\ x + y + az &= b \end{aligned}$$

where x, y and z are the unknowns and a and b are constants.

(a) For which values of the constants a and b is there no solution? (4 marks)

Solution
$\begin{aligned} x + 2y + 3z &= 2 \\ 3x + 7y + 11z &= 6 \\ x + y + az &= b \end{aligned} \Rightarrow \begin{aligned} x + 2y + 3z &= 2 \\ y + 2z &= 0 \\ -y + (a-3)z &= b-2 \end{aligned} \Rightarrow \begin{aligned} x + 2y + 3z &= 2 \\ y + 2z &= 0 \\ (a-1)z &= b-2 \end{aligned}$
No solution: Last equation must be inconsistent, i.e. $a = 1$ and $b \neq 2$
Specific behaviours
✓ first reduction ✓ second reduction ✓ notes inconsistency ✓ obtains $a = 1$ and $b \neq 2$

(b) Solve the equations given that $a = 5$ and $b = 3$. (3 marks)

Solution
$a = 5$ and $b = 3 \Longrightarrow \begin{aligned} x + 2y + 3z &= 2 \\ y + 2z &= 0 \\ 4z &= 1 \end{aligned}$ $z = \frac{1}{4}$ back substitution gives $y = -\frac{1}{2}$ and $x = \frac{9}{4}$
Specific behaviours
✓ obtains $z = \frac{1}{4}$ ✓ back substitutes to obtain $y = -\frac{1}{2}$ ✓ back substitutes to obtain $x = \frac{9}{4}$

(c) For which values of the constants a and b are there precisely two solutions? (1 mark)

Solution
Precisely two solutions: Never If there is more than one solution there are infinitely many
Specific behaviours
✓ correct answer

Question 6 **(6 marks)**

The Cartesian equation of a sphere $\mathcal{S}$ is

$$x^2 + y^2 + z^2 = 2x + 4y - 4z$$

(a) By rearranging the equation in the form $(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$, determine the coordinates of the centre *C* of $\mathcal{S}$ and its radius. (2 marks)

Solution
$x^2 + y^2 + z^2 = 2x + 4y - 4z \Longrightarrow x^2 - 2x + y^2 - 4y + z^2 + 4z = 0$ $\Longrightarrow (x-1)^2 + (y-2)^2 + (z+2)^2 = 1 + 4 + 4 = 9$ So (1,2,-2) are the coordinates of the centre *C* and the radius is 3
Specific behaviours
✓ obtains coordinates of *C* ✓ obtains radius r

(b) Show that the origin *O* lies on $\mathcal{S}$. (1 mark)

Solution
$(x-1)^2 + (y-2)^2 + (z+2)^2 = 9$ Subs (0,0,0) into equation $(0-1)^2 + (0-2)^2 + (0+2)^2 = 9$
Specific behaviours
✓ correct answer

(c) Find the coordinates of the point *A* on $\mathcal{S}$ that is diametrically opposite to *O*. (1 mark)

Solution
$\overrightarrow{OP} = 2\overrightarrow{OC}$ so the coordinates of *A* are (2,4,-4)
Specific behaviours
✓ correct answer

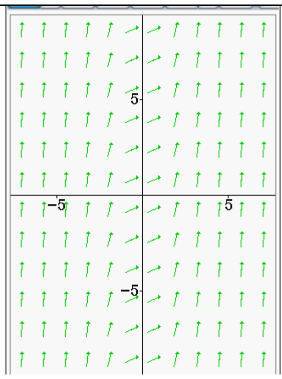
(d) Find the Cartesian equation of the plane $\mathcal{P}$ which contains the point *A* and is tangent to $\mathcal{S}$.
Hint: The radial vector $\overrightarrow{OC}$ is normal to $\mathcal{P}$. (2 marks)

Solution
Using $\boldsymbol{r} \cdot \boldsymbol{n} = c$ with $\boldsymbol{n} = \overrightarrow{\boldsymbol{OC}}$ and $c = \overrightarrow{\boldsymbol{OA}} \cdot \boldsymbol{n}$ gives $x + 2y - 2z = 2 \times 1 + 4 \times 2 + (-4) \times (-2) = 18$
Specific behaviours
✓ uses $\boldsymbol{r} \cdot \boldsymbol{n} = c$ with $\boldsymbol{n} = \overrightarrow{\boldsymbol{OC}}$ and $c = \overrightarrow{\boldsymbol{OA}} \cdot \boldsymbol{n}$ ✓ simplifies

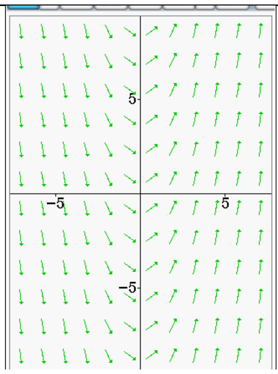
Question 7 **(8 marks)**

(a) From the differential equations provided, select and state the one that matches each respective slope field drawn below. $y' = x$, $y' = x^2$, $y' = 8 - 4x$, $y' = \frac{1}{x}$

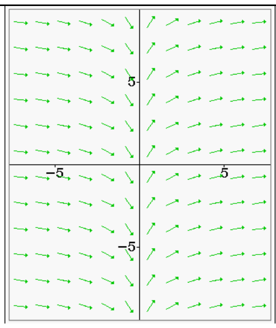
i)

Solution
 $y' = x^2$
✓ selects the correct differential equation

ii)

Solution
 $y' = x$
✓ selects the correct differential equation

iii)

Solution
 $y' = \frac{1}{x}$
✓ selects the correct differential equation

b) Consider the slope field for $y' = \dfrac{x}{y}$

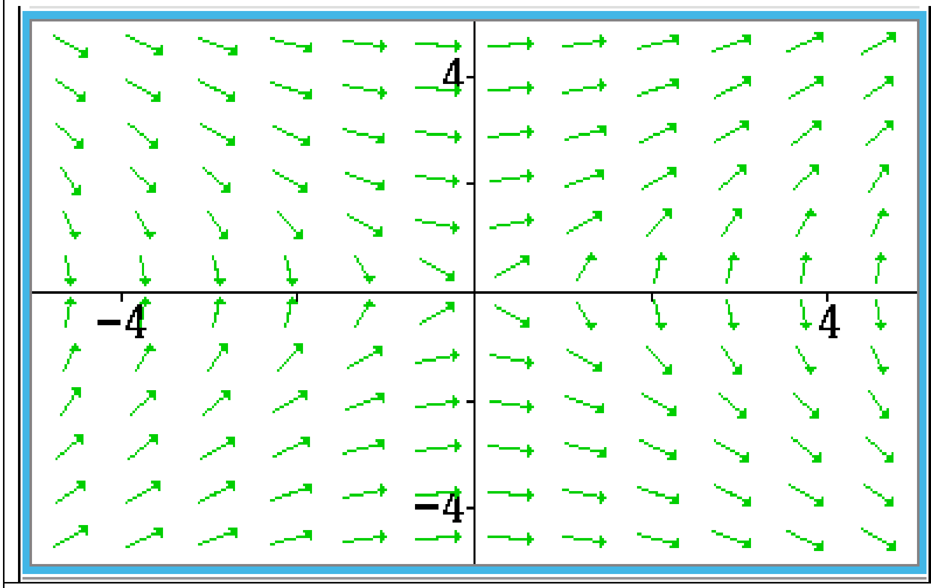
i)For what values of x and y will $y' = 0$?

Solution
$x = 0$ excluding $y = 0$
✓ states $x = 0$ and excludes $y = 0$

ii) For what values of x and y will $y' = 1$?

Solution
$y = x$ excluding (0,0)
✓ states $y = x$

iii) On the axes below, sketch the slope field for $y' = \dfrac{x}{y}$

Solution

✓ shows that slope is one along $y = x$ ✓ shows that slope is positive for quadrants 1 and 3 ✓ shows that slope is greater than one between $y = x$ and x axis.